

# SHARP EIGENVALUE BOUNDS ON QUANTUM STAR GRAPHS

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**ABSTRACT.** We prove that the optimal constant in the Lieb–Thirring inequality on a star graph with  $N$  edges coincides with that on  $\mathbb{R}$  if  $N$  is even. For odd  $N$  we show that this property holds when restricting to radial potentials and we prove an almost optimal bound for general potentials.

## 1. INTRODUCTION

Recently there has been a lot of activity in a mathematical understanding of quantum graphs, which appear as idealized models of linear, network-shaped structures in mesoscopic physics. A large literature on the subject has arisen and we refer, for instance, to the bibliography given in [1, 5] and the textbook [2]. In particular, in the papers [3, 4, 6, 7] bounds we derived on the discrete eigenvalues of Schrödinger operators on metric graphs. In the present paper we will be interested in *optimal constants* in such bounds for one of the simplest classes of metric graphs, namely *star graphs*. By  $\Gamma_N$  we denote  $N$  half-lines  $[0, \infty)$  with their endpoints 0 identified. Thus,  $\Gamma_N$  is a graph with a single vertex and  $N$  edges.

We consider the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V \quad \text{in } L_2(\Gamma_N)$$

with a potential  $V : \Gamma_N \rightarrow \mathbb{R}$ . It is well-known that, if  $V_- \in L_p(\Gamma_N)$  for some  $p \geq 1$  and  $V_+ \in L_1^{\text{loc}}(\Gamma_N)$ , then the Schrödinger operator can be defined as a self-adjoint operator in  $L_2(\Gamma_N)$  via the lower semi-bounded and closed quadratic form

$$h[\psi] := \int_{\Gamma_N} (|\psi'|^2 + V|\psi|^2) dx, \quad \psi \in H^1(\Gamma_N) \cap L_2(\Gamma_N, V_+ dx).$$

By definition, a function  $\psi$  on  $\Gamma_N$  belongs to the Sobolev space  $H^1(\Gamma_N)$  if its  $N$  restrictions  $\psi_1, \dots, \psi_N$  to the edges of  $\Gamma_N$  belong to  $H^1(0, \infty)$  and if their values at the vertex coincide. This definition of the Schrödinger operator via quadratic forms gives rise, in a generalized sense, to the so-called *Kirchhoff boundary conditions* at the vertex,

$$\sum_{j=1}^N \psi_j'(0+) = 0.$$

Moreover, the condition  $V_- \in L_p(\Gamma_N)$  with  $p < \infty$  guarantees that the negative spectrum of the Schrödinger operator consists of discrete eigenvalues of finite multiplicities.

As usual, we write  $\text{Tr } H_-^\gamma$  for the sum of the  $\gamma$ -th power of the absolute values of the negative eigenvalues of  $H$ .

One can prove [4] that for any  $\gamma \geq 1/2$  there is a constant  $L_{\gamma,N}$  such that

$$\text{Tr } H_-^\gamma \leq L_{\gamma,N} \int_{\Gamma_N} V_-^{\gamma+1/2} dx. \quad (1)$$

In the following, we will denote by  $L_{\gamma,N}$  the *optimal* (that is, smallest possible) value of the constant in (1). We are interested in characterizing this value and, in particular, in relating it to  $L_{\gamma,2} =: L_\gamma$  for  $\Gamma_2 = \mathbb{R}$ , that is, the optimal constant in the inequality

$$\text{Tr} \left( -\frac{d^2}{dx^2} + V \right)_-^\gamma \leq L_\gamma \int_{\mathbb{R}} V_-^{\gamma+1/2} dx. \quad (2)$$

Finding the optimal constant in (2) is a famous open problem due to Lieb and Thirring [11]. What is currently known is that

$$L_{1/2} = 1/4 \quad \text{and} \quad L_\gamma = (4\pi)^{-1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+3/2)} \text{ if } \gamma \geq 3/2; \quad (3)$$

see [8, 11] and also [9, 10] for a review and results in higher dimensions.

By taking a compactly supported almost-optimal potential for (2) and transplanting it very far out on a single edge of  $\Gamma_N$  it is easy to see that

$$L_{\gamma,N} \geq L_\gamma \quad \text{for all } \gamma \geq 1/2 \text{ and all } N \in \mathbb{N}. \quad (4)$$

Thus, in the following we will be interested in *upper bounds* on  $L_{\gamma,N}$ .

In [3] we have shown that

$$L_{\gamma,N} = L_\gamma \quad \text{for all } \gamma \geq 2 \text{ and all } N \in \mathbb{N}. \quad (5)$$

In fact, this equality is valid for a large number of graphs, but, remarkably, not for all graphs; see [3] for an explicit counterexample. As far as we know, there are no optimal results on Lieb–Thirring constants on quantum graphs apart from (5). We emphasize that the proof in [3] proceeds by showing  $L_{\gamma,N} \leq (4\pi)^{-1/2} \Gamma(\gamma+1)/\Gamma(\gamma+3/2)$  directly, without comparing  $L_{\gamma,N}$  to  $L_\gamma$ .

In this paper we shall do exactly the latter, namely, we find a comparison method to relate  $L_{\gamma,N}$  to  $L_\gamma$ , without needing to know the explicit value of  $L_\gamma$ . This allows us to settle the problem completely for even  $N$  as well as, under a symmetry assumption, for odd  $N$ . The following two theorems are our main results.

**Theorem 1.** *Let  $\gamma \geq 1/2$ . If  $N$  is even, then*

$$L_{\gamma,N} = L_\gamma$$

*and, if  $N$  is odd, then*

$$L_{\gamma,N} \leq \frac{N+1}{N} L_\gamma,$$

*where  $L_{\gamma,N}$  and  $L_\gamma$  are the optimal constants in (1) and (2), respectively.*

*Remarks.* (1) For even  $N$ , this theorem together with (3) yields explicitly the optimal constant for  $\gamma = 1/2$  and  $\gamma \geq 3/2$ . This improves our earlier bound from [3] for  $\gamma \geq 2$ . We emphasize that none of the methods used to prove (3) seem to generalize in an obvious way to  $\Gamma_N$ .

(2) A variant of our proof shows that if  $L_{\gamma, N_0} = L_\gamma$  for some odd  $N_0$ , then  $L_{\gamma, N} = L_\gamma$  for all  $N \geq N_0$ ; see Proposition 4.

(3) For  $N = 1$ , our bound states  $L_{\gamma, 1} \leq 2L_\gamma$ . The proof of Lemma 3 shows that this bound is optimal as long as the optimal potential for  $L_\gamma$  has a single bound state. This holds, in particular, for  $\gamma = 1/2$ .

(4) For odd  $N \geq 3$  our bound uses the bound  $L_{\gamma, 1} \leq 2L_\gamma$  for  $N = 1$ . If the latter bound can be improved for some (large)  $\gamma$ , then also our bounds for arbitrary odd  $N \geq 3$  improve automatically.

We call a function  $V$  on  $\Gamma_N$  *radial* if the value of  $V(x)$  depends only on the distance of  $x$  from the vertex of  $\Gamma_N$ . Let us denote by  $L_{\gamma, N}^{(\text{rad})}$  the optimal constant in (1) *when restricted to radial functions*  $V$ .

**Theorem 2.** *Let  $\gamma \geq 1/2$ . For any  $N \geq 2$ ,*

$$L_{\gamma, N}^{(\text{rad})} = L_\gamma,$$

*where  $L_{\gamma, N}^{(\text{rad})}$  is the optimal constant in the radial version of (1) and  $L_\gamma$  is the optimal constant in (2).*

We will prove Theorem 1 in Section 2 and Theorem 2 in Section 3.

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## 2. PROOF OF THEOREM 1

We begin with the proof of Theorem 1 for  $N = 1$ . This is the following bound on the eigenvalues of a half-line Schrödinger operator with Neumann boundary conditions. More precisely, this operator is defined via the quadratic form  $\int_{\mathbb{R}_+} (|\psi'|^2 + V|\psi|^2) dx$  in  $L_2(\mathbb{R}_+)$  with form domain  $H^1(\mathbb{R}_+) \cap L_2(\mathbb{R}_+, V_+ dx)$ .

**Lemma 3.** *Let  $H^{(\text{Neu})} = -\frac{d^2}{dx^2} + V$  in  $L_2(\mathbb{R}_+)$  with Neumann boundary conditions. Then, for all  $\gamma \geq 1/2$ ,*

$$\text{Tr} \left( H^{(\text{Neu})} \right)_-^\gamma \leq 2L_\gamma \int_{\mathbb{R}_+} V_-^{\gamma+1/2} dx.$$

*Proof.* We extend  $V$  to a symmetric function  $\tilde{V}$  on  $\mathbb{R}$  and obtain, by the same arguments as in the proof of Theorem 2 below,

$$\text{Tr} \left( H^{(\text{Neu})} \right)_-^\gamma + \text{Tr} \left( H^{(\text{Dir})} \right)_-^\gamma = \text{Tr} \left( H^{\mathbb{R}} \right)_-^\gamma,$$

where  $H^{(\text{Dir})}$  is the same as  $H^{(\text{Neu})}$  but with Dirichlet boundary conditions and  $H^{\mathbb{R}}$  is the operator  $-\frac{d^2}{dx^2} + \tilde{V}$  in  $L_2(\mathbb{R})$ . The claimed bound follows from the inequalities  $\text{Tr}(H^{(\text{Dir})})_-^\gamma \geq 0$  and (2), that is,

$$\text{Tr}(H^{\mathbb{R}})_-^\gamma \leq L_\gamma \int_{\mathbb{R}} \tilde{V}_-^{\gamma+1/2} dx = 2L_\gamma \int_{\mathbb{R}_+} V_-^{\gamma+1/2} dx. \quad \square$$

We now turn to star graphs  $\Gamma_N$  with  $N \geq 3$ . Lower bounds on the eigenvalues can be obtained by decoupling the edges. If we would decouple all the edges, we would end up with  $N$  half-line Schrödinger operators with Neumann boundary conditions. Applying Lemma 3 to each of these operators we would obtain the bound  $L_{\gamma,N} \leq 2L_\gamma$ , which is not optimal. The idea in the following proof is to apply a more subtle decoupling.

*Proof of Theorem 1. Case  $N$  even.* We write  $N = 2n$  and consider the quadratic form  $h^{(\text{cut})}[\psi]$ , given by the same expression as  $h[\psi]$ , but with form domain

$$\{\psi \in L_2(\Gamma_N) : \forall 1 \leq j \leq N : \psi_j \in H^1(\mathbb{R}_+) \text{ and } \forall 1 \leq j \leq n : \psi_j(0) = \psi_{j+N}(0)\}.$$

In other words, we decompose  $\Gamma_N$  into  $n$  copies of  $\mathbb{R}$ , namely,  $e_1 \cup e_{n+1}, \dots, e_n \cup e_N$ , where  $e_1, \dots, e_N$  are the edges of  $\Gamma_N$ . Since the form domain of  $h^{(\text{cut})}$  contains that of  $h$ , the corresponding operator  $H^{(\text{cut})}$  satisfies  $H^{(\text{cut})} \leq H$  in the sense of quadratic forms, and therefore

$$\text{Tr} H_-^\gamma \leq \text{Tr}(H^{(\text{cut})})_-^\gamma \quad (6)$$

for any  $\gamma$ . Since for the operator  $H^{(\text{cut})}$  each edge is only connected to one other edge, we have

$$H^{(\text{cut})} \sim \bigoplus_{i=1}^n H_i,$$

where  $H_i$  is the Schrödinger operator in  $L_2(\mathbb{R})$  with potential  $\tilde{V}_i$  given for  $t > 0$  by

$$\tilde{V}_i(t) = V_i(t), \quad \tilde{V}_i(-t) = V_{n+i}(t).$$

(Here,  $V_i$  and  $V_{n+i}$  denote the restrictions of  $V$  to the  $i$ -th and  $n+i$ -th edge.) Thus,

$$\text{Tr}(H^{(\text{cut})})_-^\gamma = \sum_{i=1}^n \text{Tr}(H_i)_-^\gamma. \quad (7)$$

Finally, if  $\gamma \geq 1/2$ , we can use the Lieb–Thirring inequality (2) to bound

$$\text{Tr}(H_i)_-^\gamma \leq L_\gamma \int_{\mathbb{R}} (\tilde{V}_i)_-^{\gamma+1/2} dt. \quad (8)$$

Combining (6), (7) and (8) we obtain for  $\gamma \geq 1/2$

$$\text{Tr} H_-^\gamma \leq L_\gamma \sum_{i=1}^n \int_{\mathbb{R}} (\tilde{V}_i)_-^{\gamma+1/2} dt = L_\gamma \int_{\Gamma_N} V_-^{\gamma+1/2} dx,$$

as claimed.

*Case  $N$  odd.* We shall show that for  $\gamma \geq 1/2$ ,

$$\mathrm{Tr} H_-^\gamma \leq L_\gamma \int_{\Gamma_N} V_-^{\gamma+1/2} dx + L_\gamma \int_{\mathbb{R}_+} (V_N)_-^{\gamma+1/2} dt. \quad (9)$$

After relabelling the edges this yields

$$\mathrm{Tr} H_-^\gamma \leq L_\gamma \int_{\Gamma_N} V_-^{\gamma+1/2} dx + L_\gamma \int_{\mathbb{R}_+} (V_i)_-^{\gamma+1/2} dt$$

for any  $i = 1, \dots, N$ , and summing this inequality over  $i$ , we obtain

$$N \mathrm{Tr} H_-^\gamma \leq (N+1) L_\gamma \int_{\Gamma_N} V_-^{\gamma+1/2} dx,$$

which is the claimed inequality.

Thus, it remains to prove (9). This time we define a quadratic form  $h^{(\mathrm{cut})}$  by the same expression as  $h[\psi]$  but with form domain

$$\{\psi \in L_2(\Gamma_N) : \forall 1 \leq j \leq N : \psi_j \in H^1(\mathbb{R}_+) \text{ and } \psi_1(0) = \dots = \psi_{N-1}(0)\}.$$

As before, we have (6). Since the  $N$ -th edge is disconnected from the rest of the edges, we have

$$H^{(\mathrm{cut})} \sim \tilde{H} \oplus \tilde{H}_N,$$

where  $\tilde{H}$  is the operator in  $L_2(\Gamma_{N-1})$ , which is obtained by ignoring the  $N$ -th edge, and  $\tilde{H}_N$  is the Schrödinger operator in  $L_2(\mathbb{R}_+)$  with potential  $V_N$  and a Neumann boundary condition. Thus,

$$\mathrm{Tr}(H^{(\mathrm{cut})})_-^\gamma = \mathrm{Tr} \tilde{H}_-^\gamma + \mathrm{Tr}(\tilde{H}_N)_-^\gamma. \quad (10)$$

Since  $N-1$  is even, we have according to Step 1

$$\mathrm{Tr} \tilde{H}_-^\gamma \leq L_\gamma \sum_{i=1}^{N-1} \int_{\mathbb{R}_+} (V_i)_-^{\gamma+1/2} dt. \quad (11)$$

On the other hand, by Lemma 3,

$$\mathrm{Tr}(\tilde{H}_N)_-^\gamma \leq 2L_\gamma \int_{\mathbb{R}_+} (V_N)_-^{\gamma+1/2} dt. \quad (12)$$

The claimed inequality (9) now follows from (6), (10), (11) and (12). This concludes the proof of the theorem.  $\square$

A refinement of the previous proof yields

**Proposition 4.** *Let  $\gamma \geq 1/2$ . If  $N_0 < N$  are both odd, then*

$$L_{\gamma,N} \leq ((N - N_0)/N) L_\gamma + (N_0/N) L_{\gamma,N_0}.$$

*In particular, if  $L_{\gamma,N_0} = L_\gamma$  for some odd  $N_0 \in \mathbb{N}$ , then  $L_{\gamma,N} = L_\gamma$  for all  $N \geq N_0$ .*

Note that the bound in Theorem 1 follows by taking  $N_0 = 1$  and using  $L_{\gamma,1} \leq 2L_\gamma$  according to Lemma 3.

*Proof.* We argue as in the odd  $N$  case of Theorem 1 and decouple  $\Gamma_N$  into two star graphs  $\Gamma_{N_0}$  and  $\Gamma_{N-N_0}$ . For  $\Gamma_{N_0}$  we use the bound with  $L_{\gamma, N_0}$  and for  $\Gamma_{N-N_0}$  we use the bound with  $L_\gamma$  (since  $N - N_0$  is even). Finally, we sum over all possible choices of  $N_0$  edges, as in the equations after (9). We omit the details.  $\square$

### 3. PROOF OF THEOREM 2

We now turn our attention to radial potentials  $V$  on  $\Gamma_N$  and show that the constant  $L_{\gamma, N}^{(\text{rad})}$  coincides with the optimal one-dimensional constant  $L_\gamma$ . This holds both for even and odd  $N$ .

The symmetry of  $\Gamma_N$  allows one to construct an orthogonal decomposition of the space  $L_2(\Gamma_N)$  which reduces the Kirchhoff Laplacian. If, in addition,  $V$  is radial, it also reduces the operator  $H$ . The study of the spectrum of  $H$  is then reduced to the study of the spectrum of the orthogonal components in the decomposition, where each component can be identified with a Schrödinger operator acting in the space  $L_2(\mathbb{R}_+)$ .

In [7, 12, 13] a decomposition of the  $L_2$  space was given for so-called regular, rooted metric trees. In what follows, we reformulate the decomposition of  $L_2(\Gamma_N)$  for our purposes. We denote by  $\mathcal{H}^{(0)}$  the closed subspace of  $L_2(\Gamma_N)$  of all radial functions on  $\Gamma_N$ , i.e.,

$$\mathcal{H}^{(0)} := \{\psi \in L_2(\Gamma_N) : \forall r \geq 0 : \psi_1(r) = \psi_2(r) = \dots = \psi_N(r)\},$$

where  $\psi_j := \psi|_{e_j}$ . Any radial function  $\psi$  on  $\Gamma_N$  can be identified with the function  $s := R\psi$  on the half-line  $[0, \infty)$  such that  $\psi(x) = s(|x|)$  for each  $x \in \Gamma_N$ , and

$$\int_{\Gamma_N} |\psi(x)|^2 dx = N \int_0^\infty |s(x)|^2 dx, \quad \psi \in \mathcal{H}^{(0)}, s = R\psi.$$

Thus, the operator  $\sqrt{N}R$  defines an isometry of the subspace  $\mathcal{H}^{(0)}$  onto the space  $L_2(\mathbb{R}_+)$ .

To state the orthogonal decomposition of  $L_2(\Gamma_N)$  we define for  $1 \leq \ell \leq N-1$ , the following orthogonal subspaces,

$$\mathcal{H}^{(\ell)} := \{\psi \in L_2(\Gamma_N) : \forall j = 1, \dots, N, \forall r \geq 0 : \psi_{j+1}(r) = e^{i2\pi(\ell/N)} \psi_j(r)\}.$$

(Here, we write  $\psi_{N+1} = \psi_1$ .) Clearly, as for  $\ell = 0$  there are isometries from  $\mathcal{H}^{(\ell)}$  onto  $L_2(\mathbb{R}_+)$ .

**Lemma 5.** *The subspaces  $\mathcal{H}^{(\ell)}$ ,  $\ell = 0, \dots, N-1$ , are mutually orthogonal and*

$$L_2(\Gamma_N) = \bigoplus_{\ell=0}^{N-1} \mathcal{H}^{(\ell)}. \tag{13}$$

*Proof.* First, we show that  $L_2(\Gamma_N) = \text{span} \{\mathcal{H}^{(\ell)} : \ell\}$ , i.e., for every function  $\psi \in L_2(\Gamma_N)$  there are functions  $\psi^{(\ell)} \in \mathcal{H}^{(\ell)}$  such that  $\psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)}$ . (Note that for  $N = 2$  this corresponds to the fact that every function on the real line is given as a sum of even and odd functions.)

We can write  $\psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)}$ , where the functions  $\psi^{(\ell)}$  are defined via their restrictions  $\psi_k^{(\ell)}$  to the  $k$ -th edge,  $k = 1, \dots, N$ , by

$$\psi_k^{(0)}(t) = \frac{1}{N} \sum_{j=1}^N \psi_j(t)$$

and, for  $\ell = 1, \dots, N-1$ ,

$$\psi_k^{(\ell)} = \frac{1}{N} \left( \psi_k(t) + \sum_{j \neq k} e^{i2\pi\ell/N} \psi_j(t) \right).$$

The identity  $\psi = \sum_{\ell=0}^{N-1} \psi^{(\ell)}$  follows from the fact that

$$\sum_{\ell=0}^{N-1} e^{i2\pi\ell/N} = \sum_{\ell=0}^{N-1} (e^{i2\pi/N})^\ell = \frac{(e^{i2\pi/N})^N - 1}{(e^{i2\pi/N}) - 1} = 0.$$

Moreover, it is easy to verify that  $\psi^{(\ell)} \in \mathcal{H}^{(\ell)}$ .

To prove the lemma, it remains to show that the spaces  $\mathcal{H}^{(\ell)}$ ,  $0 \leq \ell \leq N-1$ , are mutually orthogonal. For  $\psi^{(\ell)} \in \mathcal{H}^{(\ell)}$  and  $\psi^{(m)} \in \mathcal{H}^{(m)}$  with  $\ell \neq m$  consider

$$\begin{aligned} \int_{\Gamma} \psi^{(\ell)} \overline{\psi^{(m)}} dx &= \sum_{j=1}^N \int_{\mathbb{R}_+} \psi_j^{(\ell)} \overline{\psi_j^{(m)}} dt = \sum_{j=1}^N \int_{\mathbb{R}_+} e^{2i\pi\ell(j-1)/N} \psi_1^{(\ell)} e^{-2i\pi m(j-1)/N} \overline{\psi_1^{(m)}} dt \\ &= \int_{\mathbb{R}_+} \psi_1^{(\ell)} \overline{\psi_1^{(m)}} dt \sum_{j=1}^N (e^{2i\pi(\ell-m)/N})^{j-1}. \end{aligned}$$

The right-hand side equals zero since

$$\sum_{j=1}^N (e^{2i\pi(\ell-m)/N})^{j-1} = \sum_{j=0}^{N-1} (e^{2i\pi(\ell-m)/N})^j = \frac{(e^{2i\pi(\ell-m)/N})^N - 1}{(e^{2i\pi(\ell-m)/N}) - 1} = 0.$$

Hence, the spaces  $\mathcal{H}^{(\ell)}$ ,  $0 \leq \ell \leq N-1$ , are mutually orthogonal, as claimed.  $\square$

A function in  $\mathcal{H}^{(\ell)}$  is completely determined by its restriction to one of the edges. We now characterize the  $H^1(\Gamma_N)$  property of a function in  $\mathcal{H}^{(\ell)}$  in terms of its restrictions. Clearly, a function in  $\mathcal{H}^{(0)}$  belongs to  $H^1(\Gamma_N)$  iff its restrictions belong to  $H^1(\mathbb{R}_+)$ . On the other hand, a function  $\psi \in \mathcal{H}^{(\ell)}$  with  $\ell = 1, \dots, N-1$  belongs to  $H^1(\Gamma_N)$  iff its restrictions belong to  $H^{1,0}(\mathbb{R}_+) = \{\psi \in H^1(\mathbb{R}_+) : \psi(0) = 0\}$ . The crucial point here is the Dirichlet boundary condition at the origin. Moreover, we have

$$\int_{\Gamma_N} |\psi'|^2 dx = \sum_{\ell=0}^{N-1} \int_{\Gamma_N} |(\psi^{(\ell)})'|^2 dx,$$

where  $\psi^{(\ell)}$  denotes the projection of  $\psi$  onto  $\mathcal{H}^{(\ell)}$ .

We conclude that the subspaces  $\mathcal{H}^{(\ell)}$  reduce the Schrödinger operator  $H$  and that the operators  $H|_{\mathcal{H}^{(\ell)}}$  are unitarily equivalent to operators  $H^{(\ell)}$  in  $L_2(\mathbb{R}_+)$ . These operators act as  $-\frac{d^2}{dx^2} + V(x)$  and have Neumann (if  $\ell = 0$ ) and Dirichlet (if  $\ell = 1, \dots, N-1$ )

boundary conditions. Here we identify the radial function  $V$  on  $\Gamma_N$  in a natural way with a function on  $\mathbb{R}_+$ . (More precisely, the operators  $H^{(\ell)}$  are defined via the quadratic form  $\int_{\mathbb{R}_+} (|\psi'|^2 + V|\psi|^2) dx$  with form domain  $H^1(\mathbb{R}_+)$  for  $\ell = 0$  and  $H^{1,0}(\mathbb{R}_+)$  for  $\ell = 1, \dots, N-1$ .) Clearly, the operators  $H^{(\ell)}$  for  $\ell = 1, \dots, N-1$  coincide.

To summarize, the operator  $H$  in  $L_2(\Gamma_N)$  is unitary equivalent to the orthogonal sum of the operators  $H^{(\ell)}$  on  $L_2(\mathbb{R}_+)$ ,

$$H \sim \bigoplus_{\ell=0}^{N-1} H^{(\ell)}, \quad (14)$$

and therefore its eigenvalues, counting multiplicities, are given by the union of the eigenvalues of  $H^{(\ell)}$ , counting multiplicities. Then, for any  $\gamma$ ,

$$\mathrm{Tr} H_-^\gamma = \mathrm{Tr} (H^{(0)})_-^\gamma + (N-1) \mathrm{Tr} (H^{(1)})_-^\gamma. \quad (15)$$

Consider now the Schrödinger operator

$$\tilde{H} = -\frac{d^2}{dx^2} + \tilde{V} \quad \text{in } L_2(\mathbb{R}),$$

where the potential  $\tilde{V}$  is the symmetric extension of the potential  $V|_{e_j}$  to the whole line. The unitary equivalence (14) with  $N = 2$  implies that  $\tilde{H} \sim H^{(0)} \oplus H^{(1)}$ . Reinserting this into (14) we find

$$H \sim \tilde{H} \oplus \bigoplus_{\ell=2}^{N-1} H^{(\ell)},$$

and hence

$$\mathrm{Tr} H_-^\gamma = \mathrm{Tr} (\tilde{H})_-^\gamma + (N-2) \mathrm{Tr} (H^{(1)})_-^\gamma. \quad (16)$$

This is the key identity in the radial case.

According to the Lieb–Thirring inequality (2), for the first trace on the right side of (16) and  $\gamma \geq 1/2$  we have

$$\mathrm{Tr} (\tilde{H})_-^\gamma \leq L_\gamma \int_{\mathbb{R}} (\tilde{V})_-^{\gamma+1/2} dx = 2L_\gamma \int_{\mathbb{R}_+} V_-^{\gamma+1/2} dx.$$

On the other hand, by the variational principle, inequality (2) remains also true for the eigenvalues of Dirichlet half-line operators, and therefore for the second trace on the right side of (16) we have

$$\mathrm{Tr} (H^{(1)})_-^\gamma \leq L_\gamma \int_{\mathbb{R}_+} V_-^{\gamma+1/2} dx.$$

Thus, the right side of (16) is bounded from above by

$$NL_\gamma \int_{\mathbb{R}_+} V_-^{\gamma+1/2} dx = L_\gamma \int_{\Gamma_N} V_-^{\gamma+1/2} dx,$$

which proves the bound  $L_{\gamma,N}^{(\mathrm{rad})} \leq L_\gamma$  claimed in Theorem 2.

Conversely, for any  $\varepsilon > 0$  there is a compactly supported  $V$  on  $\mathbb{R}$  such that

$$\mathrm{Tr} \left( -\frac{d^2}{dx^2} + V \right)_-^\gamma \geq (1 - \varepsilon) L_\gamma \int_{\mathbb{R}} V_-^{\gamma+1/2} dx. \quad (17)$$

We denote by  $V_a(x) = V(x - a)$  the translate of this potential and choose  $a$  so large that the support of  $V_a$  is contained in  $\mathbb{R}_+$ . We use  $V_a$  as a radial potential on  $\Gamma_N$  and denote the corresponding operator by  $H_a$  and its parts on  $\mathcal{H}^{(0)}$  and  $\mathcal{H}^{(1)}$  by  $H_a^{(0)}$  and  $H_a^{(1)}$ , respectively. It is easy to see that as  $a \rightarrow \infty$ ,

$$\frac{\mathrm{Tr} \left( H_a^{(0)} \right)_-^\gamma}{\mathrm{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma} \rightarrow 1 \quad \text{and} \quad \frac{\mathrm{Tr} \left( H_a^{(1)} \right)_-^\gamma}{\mathrm{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma} \rightarrow 1.$$

On the other hand, by translation invariance,  $\mathrm{Tr} \left( -\frac{d^2}{dx^2} + V_a \right)_-^\gamma = \mathrm{Tr} \left( -\frac{d^2}{dx^2} + V \right)_-^\gamma$  and  $\int_{\Gamma_N} (V_a)_-^{\gamma+1/2} dx = N \int_{\mathbb{R}} V_-^{\gamma+1/2} dx$ . Therefore (15) and (17) yield

$$\liminf_{a \rightarrow \infty} \frac{\mathrm{Tr} (H_a)_-^\gamma}{\int_{\Gamma_N} (V_a)_-^{\gamma+1/2} dx} \geq (1 - \varepsilon) L_\gamma.$$

This proves  $L_{\gamma,N}^{(\mathrm{rad})} \geq (1 - \varepsilon) L_\gamma$  and, since  $\varepsilon > 0$  is arbitrary, we obtain  $L_{\gamma,N}^{(\mathrm{rad})} \geq L_\gamma$ . This concludes the proof of the theorem.  $\square$

## REFERENCES

- [1] Gregory Berkolaiko, Robert Carlson, Stephen A. Fulling, and Peter Kuchment, editors. *Quantum Graphs and Their Applications*, volume 415 of *Contemporary Mathematics*, Providence, RI, 2006. American Mathematical Society.
- [2] Gregory Berkolaiko and Peter Kuchment. *Introduction to quantum graphs*, volume 186 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.
- [3] Semra Demirel and Evans M. Harell II. On semiclassical and universal inequalities for eigenvalues of quantum graphs. *Rev. Math. Phys.*, 22:305–329, 2010.
- [4] Tomas Ekholm, Rupert L. Frank, and Hynek Kovařík. Eigenvalue estimates for Schrödinger operators on metric trees. *Adv. Math.*, 226(6):5165–5197, 2011.
- [5] Pavel Exner, Jonathan P. Keating, Peter Kuchment, Toshikazu Sunada, and Alexander Teplyaev, editors. *Analysis on graphs and its applications*, volume 77 of *Proceedings of Symposia in Pure Mathematics*. American Mathematical Society, Providence, RI, 2008. Papers from the program held in Cambridge, January 8–June 29, 2007.
- [6] Pavel Exner, Ari Laptev, and Muhammad Usman. On some sharp spectral inequalities for Schrödinger operators on semiaxis. *Comm. Math. Phys.*, 326(2):531–541, 2014.
- [7] Rupert L. Frank and Hynek Kovařík. Heat kernels of metric trees and applications. *SIAM J. Math. Anal.*, 45(3):1027–1046, 2013.
- [8] Dirk Hundertmark, Elliott H. Lieb, and Lawrence E. Thomas. A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator. *Adv. Theor. Math. Phys.*, 2(4):719–731, 1998.
- [9] Norman E. Hurt. *Mathematical physics of quantum wires and devices*, volume 506 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2000. From spectral resonances to Anderson localization.

- [10] Ari Laptev and Timo Weidl. Sharp Lieb-Thirring inequalities in high dimensions. *Acta Math.*, 184(1):87–111, 2000.
- [11] Elliott H. Lieb and Walter Thirring. Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities. *Studies in Mathematical Physics, Princeton University Press, Princeton, NJ*, pages 269–303, 1976.
- [12] K. Naimark and M. Solomyak. Eigenvalue estimates for the weighted Laplacian on metric trees. *Proc. London Math. Soc. (3)*, 80(3):690–724, 2000.
- [13] Michael Solomyak. On the spectrum of the Laplacian on regular metric trees. *Waves Random Media*, 14(1):S155–S171, 2004. Special section on quantum graphs.

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